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Dynamic connections in analytical mechanics

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Abstract. It is shown that any dynamic equation on a configuration bundle $Q \to \mathbf{R}$ of non-relativistic time-dependent mechanics is associated with connections on the affine jet bundle $J^1Q \to Q$ and on the tangent bundle $TQ \to Q$. As a consequence, every non-relativistic dynamic equation can be seen as a geodesic equation with respect to a (non-linear) connection on the tangent bundle $TQ \to Q$. Using this fact, the relationship between relativistic and non-relativistic equations of motion is studied. The geometric notions of reference frames and relative accelerations in non-relativistic mechanics are phrased in the terms of connections. The covariant form of non-relativistic dynamic equations is written.

1 Introduction

We are concerned with non-relativistic time-dependent mechanics whose configuration space is a bundle $Q \to \mathbf{R}$ with an m-dimensional typical fibre M over a 1-dimensional base $\mathbf{R}$, treated as a time axis. This configuration space is provided with bundle coordinates (t, q^i). The corresponding velocity phase space is the first order jet manifold J^1Q of sections of the bundle $Q \to \mathbf{R}$ [2-6,9]. It is coordinated by (t, q^i, q^i_t).

As is well known, a second order dynamic equation on a bundle $Q \to \mathbf{R}$ is defined as a first order dynamic equation on the jet manifold J^1Q, given by a holonomic connection ξ on $J^1Q \to \mathbf{R}$. The fact that ξ is a holonomic curvature-free connection places a limit on the geometric analysis of dynamic equations.

We aim to show that every dynamic equation on a configuration space Q defines a connection γ_ξ on the affine jet bundle $J^1Q \to Q$, and *vice versa*. Then, every dynamic equation on Q can be associated with a (non-linear) connection K on the tangent bundle $TQ \to Q$, and *vice versa*. Moreover, it gives rise to an equivalent geodesic equation on TQ with respect to an above-mentioned connection K due to the canonical imbedding $J^1Q \to TQ$.

In particular, let $Q = X^4$ be a world manifold of a relativistic theory. An equation of motion of a relativistic system is a geodesic equation on the tangent bundle TX of relativistic velocities. Thus, both relativistic and non-relativistic equations of motion can be seen on the tangent bundle TX, but their solutions live in the different subbundles of

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TX. We make use of this fact in order to study the relationship between relativistic and non-relativistic equations of motion.

The geometric analysis of dynamic equations also involves the connections Γ on the bundle $Q \to \mathbf{R}$ which describe reference frames in non-relativistic mechanics [1, 3, 5, 9]. In particular, one can think of the vertical vectors $(q^i_t - \Gamma^i)\partial_i$ on $Q \to \mathbf{R}$ as being the relative velocities with respect to the reference frame Γ. The notion of a relative acceleration is more intricate. Given a dynamic equation ξ, we define a frame connection γ_Γ on $J^1Q \to Q$ and then the lift ξ_Γ of a reference frame Γ to a holonomic connection on $J^1Q \to \mathbf{R}$ such that the vertical vector field $a_\Gamma = \xi - \xi_\Gamma$ describes an observable relative acceleration (or a relative force) with respect to the reference frame Γ. Then, any dynamic equation can be written in the form, covariant under coordinate transformations,

$$\widetilde{D}_{\gamma_\Gamma} q^i_t = a^i_\Gamma$$

where $\widetilde{D}_{\gamma_\Gamma}$ is the vertical covariant differential with respect to the fame connection γ_Γ.

Throughout the article, the notation $\partial/\partial q^\lambda = \partial_\lambda$, $\partial/\partial \dot{q}^\lambda = \dot{\partial}_\lambda$ is used.

2 Fibre bundles over R

In this inerlude, we point out several important peculiarities of bundles over $\mathbf{R}$. The base $\mathbf{R}$ of $Q \to \mathbf{R}$ is parameterized by a Cartesian coordinate t with the transition functions $t' = t + \text{const}$. Hence, $\mathbf{R}$ is provided with the standard vector field ∂_t and the standard 1-form dt. The symbol dt also stands for a pull-back of dt onto Q.

Any fibre bundle over $\mathbf{R}$ is obviously trivial. Every trivialization

$$\psi : Q \cong \mathbf{R} \times M \tag{1}$$

yields the corresponding trivialization of the jet bundle

$$J^1Q \cong \mathbf{R} \times TM, \qquad \dot{q}^i = q^i_t. \tag{2}$$

There is the canonical imbedding

$$\begin{aligned} &\lambda : J^1Q \hookrightarrow TQ, \\ &\lambda : (t, q^i, q^i_t) \mapsto (t, q^i, \dot{t} = 1, \dot{q}^i = q^i_t), \qquad \lambda = d_t = \partial_t + q^i_t \partial_i, \end{aligned} \tag{3}$$

where d_t denotes the total derivative. From now on, we will identify the jet manifold J^1Q with its image in TQ.

The affine jet bundle $J^1Q \to Q$ is modelled over the vertical tangent bundle VQ of $Q \to \mathbf{R}$. As a consequence, we have the canonical splitting

$$\alpha : V_Q J^1Q \cong J^1Q \underset{Q}{\times} VQ, \qquad \alpha(\partial^t_i) = \partial_i,$$

of the vertical tangent bundle V_QJ^1Q of the affine jet bundle $J^1Q\to Q$. Then the exact sequence of vector bundles over the composite bundle $J^1Q\to Q\to \mathbf{R}$ (see (14) below) reads

$$\begin{tikzcd} 0 \arrow[r] & V_QJ^1Q \arrow[r, hook, "i"] & VJ^1Q \arrow[r, "\pi_V"] & J^1Q\underset{Q}{\times} VQ \arrow[r] \arrow[ll, bend right, "\alpha^{-1}"'] & 0. \end{tikzcd}$$

Hence, we obtain the linear endomorphism

$$\widehat{v}=i\circ\alpha^{-1}\circ\pi_V : VJ^1Q \underset{J^1Q}{\longrightarrow} VJ^1Q, \qquad \widehat{v}\circ\widehat{v}=0,$$

of the vertical tangent bundle VJ^1Q of the jet bundle $J^1Q\to\mathbf{R}$. This endomorphism can be extended to the tangent bundle TJ^1Q as follows:

$$\widehat{v}(\partial_t)=-q^i_t\partial^t_i, \qquad \widehat{v}(\partial_i)=\partial^t_i, \qquad \widehat{v}(\partial^t_i)=0. \tag{4}$$

Due to the monomorphism λ (3), any connection

$$\Gamma=dt\otimes(\partial_t+\Gamma^i\partial_i) \tag{5}$$

on a fibre bundle $Q\to\mathbf{R}$ is identified with a nowhere vanishing horizontal vector field

$$\Gamma=\partial_t+\Gamma^i\partial_i \tag{6}$$

on Q. This is the horizontal lift of the standard vector field ∂_t on $\mathbf{R}$ by means of the connection (5). Conversely, any vector field Γ on Q such that $dt\rfloor\Gamma=1$ defines a connection on $Q\to\mathbf{R}$. Accordingly, the covariant differential associated with a connection Γ on $Q\to\mathbf{R}$ takes its values into the vertical tangent bundle of $Q\to\mathbf{R}$:

$$D_\Gamma : J^1Q \underset{Q}{\longrightarrow} VQ, \qquad \dot q^i\circ D_\Gamma=q^i_t-\Gamma^i.$$

Proposition 1. [3, 9]. Each connection Γ on a bundle $Q\to\mathbf{R}$ defines an atlas of local constant trivializations of $Q\to\mathbf{R}$ such that $\Gamma=\partial_t$ with respect to the proper coordinates, and *vice versa*. In particular, there is one-to-one correspondence between the complete connections Γ on $Q\to\mathbf{R}$ and the trivializations of this bundle.

Let J^1J^1Q be the repeated jet manifold of a bundle $Q\to\mathbf{R}$. It is coordinated by $(t,q^i,q^i_t,q^i_{(t)},q^i_{tt})$. There are two affine fibrations

$$\pi_{11}: J^1J^1Q\to J^1Q, \qquad q^i_t\circ\pi_{11}=q^i_t,$$
$$J^1\pi^1_0: J^1J^1Q\to J^1Q, \qquad q^i_t\circ J^1\pi^1_0=q^i_{(t)}.$$

They are isomorphic by the automorphism k of J^1J^1Q such that

$$q^i_t\circ k=q^i_{(t)}, \qquad q^i_{(t)}\circ k=q^i_t, \qquad q^i_{tt}\circ k=q^i_{tt}. \tag{7}$$

The underlying vector bundle of the affine bundle $J^1J^1Q \to J^1Q$ is $VJ^1Q \cong J^1VQ$.

By $J^1_QJ^1Q$ is meant the first order jet manifold of the affine jet bundle $J^1Q \to Q$. The adapted coordinates on $J^1_QJ^1Q$ are $(q^\lambda, q^i_t, q^i_{\lambda t})$, where we use the compact notation $(q^{\lambda=0} = t, q^i)$.

The second order jet manifold J^2Q of a bundle $Q \to \mathbf{R}$ is coordinated by $(t, q^i, q^i_t, q^i_{tt})$. The affine bundle $J^2Q \to J^1Q$ is modelled over the vector bundle

$$J^1Q \underset{Q}{\times} VQ \to J^1Q. \tag{8}$$

There are the imbeddings

$$J^2Q \xrightarrow{\lambda_2} TJ^1Q \xrightarrow{T\lambda} V_QTQ \cong T^2Q \subset TTQ,$$

$$\lambda_2 : (t, q^i, q^i_t, q^i_{tt}) \mapsto (t, q^i, q^i_t, \dot t = 1, \dot q^i = q^i_t, \dot q^i_t = q^i_{tt}). \tag{9}$$

$$T\lambda \circ \lambda_2 : (t, q^i, q^i_t, q^i_{tt}) \mapsto (t, q^i, \dot t = \overset{\circ}{t} = 1, \dot q^i = \overset{\circ}{q}{}^i = q^i_t, \ddot t = 0, \ddot q^i = q^i_{tt}), \tag{10}$$

where $(t, q^i, \dot t, \dot q^i, \overset{\circ}{t}, \overset{\circ}{q}{}^i, \ddot t, \ddot q^i)$ are the holonomic coordinates on TTQ, V_QTQ is the vertical tangent bundle of $TQ \to Q$, and T^2Q is a subbundle of TTQ, given by the coordinate relation $\dot t = \overset{\circ}{t}$.

Due to the morphism (9), a connection ξ on the jet bundle $J^1Q \to \mathbf{R}$ is represented by a horizontal vector field on J^1Q such that $\xi \rfloor dt = 1$. A connection ξ on $J^1Q \to \mathbf{R}$ is said to be holonomic if it takes its values into J^2Q.

Any connection Γ (6) on a bundle $Q \to \mathbf{R}$ gives rise to the section $J^1\Gamma$ of the affine bundle $J^1\pi^1_0$ and, by virtue of the isomorphism k (7), to the connection

$$J^1\Gamma = \partial_t + \Gamma^i\partial_i + d_t\Gamma^i\partial^t_i \tag{11}$$

on the jet bundle $J^1Q \to \mathbf{R}$.

Here, we also summarize the relevant material on composite bundles (see [3, 8] for details). Let us consider the composite bundle

$$Y \to \Sigma \to X, \tag{12}$$

where $Y \to \Sigma$ and $\Sigma \to X$ are bundles. It is equipped with bundle coordinates $(x^\lambda, \sigma^m, y^i)$ where (x^μ, σ^m) are bundle coordinates on the bundle $\Sigma \to X$ such that the transition functions $\sigma^m \to \sigma'^m(x^\lambda, \sigma^k)$ are independent of the coordinates y^i.

Let us consider the jet manifolds $J^1\Sigma$, $J^1_\Sigma Y$ and J^1Y of the bundles $\Sigma \to X$, $Y \to \Sigma$ and $Y \to X$, respectively. They are coordinated by

$$(x^\lambda, \sigma^m, \sigma^m_\lambda), \quad (x^\lambda, \sigma^m, y^i, \widetilde{y}^i_\lambda, y^i_m), \quad (x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_\lambda).$$

We have the following canonical map [10]:

$$\rho : J^1\Sigma \underset{\Sigma}{\times} J^1_\Sigma Y \underset{Y}{\longrightarrow} J^1Y, \qquad y^i_\lambda \circ \rho = y^i_m\sigma^m_\lambda + \widetilde{y}^i_\lambda. \tag{13}$$

Given a composite bundle Y (12), we have the exact sequence

$$0 \to V_\Sigma Y \hookrightarrow VY \to Y \underset{\Sigma}{\times} V\Sigma \to 0, \tag{14}$$

where $V_\Sigma Y$ is the vertical tangent bundle of $Y \to \Sigma$. Every connection

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \widetilde{A}^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i) \tag{15}$$

on $Y \to \Sigma$ determines the splitting

$$VY = V_\Sigma Y \underset{Y}{\oplus} A_\Sigma(Y \underset{\Sigma}{\times} V\Sigma),$$
$$\dot y^i \partial_i + \dot\sigma^m \partial_m = (\dot y^i - A^i_m \dot\sigma^m)\partial_i + \dot\sigma^m(\partial_m + A^i_m \partial_i),$$

of the exact sequence (14). Using this splitting, one can construct the first order differential operator, called the vertical covariant differential,

$$\widetilde{D} : J^1Y \to T^*X \underset{Y}{\otimes} V_\Sigma Y, \quad \widetilde{D} = dx^\lambda \otimes (y^i_\lambda - \widetilde{A}^i_\lambda - A^i_m \sigma^m_\lambda)\partial_i, \tag{16}$$

on the composite bundle $Y \to X$.

Given a connection A_Σ (15) on the bundle $Y \to \Sigma$ and a connection

$$B = dx^\lambda \otimes (\partial_\lambda + B^m_\lambda \partial_m + B^i_\lambda \partial_\lambda)$$

on the composite bundle $Y \to X$, there exists another connection A'_Σ on the bundle $Y \to \Sigma$ with the components

$$A'^i_m = A^i_m, \qquad A'^i_\lambda = B^i_\lambda - A^i_m B^m_\lambda. \tag{17}$$

3 Equations on a manifold

Let N be a manifold, coordinated by (q^λ). We recall some notions.

Definition 2. A second order equation on a manifold N is said to be an image $\Xi(TN)$ of a holonomic vector field

$$\Xi = \dot q^\lambda \partial_\lambda + u^\lambda \dot\partial_\lambda$$

on the tangent bundle TN. It is a closed imbedded subbundle of $TTN \to N$, given by the coordinate conditions

$$\overset{\circ}{q}{}^\lambda = \dot q^\lambda, \qquad \ddot q^\lambda = u^\lambda(q^\mu, \dot q^\mu). \tag{18}$$

By a solution of a second order equation on N is meant a curve $c : () \to N$ whose second order tangent prolongation $\ddot c$ lives in the subbundle (18).

Given a connection

$$K = dq^\lambda \otimes (\partial_\lambda + K^\mu_\lambda \dot\partial_\mu) \tag{19}$$

on the tangent bundle $TN \to N$, let

$$\widehat{K} : TN \underset{N}{\times} TN \to TTN \tag{20}$$

be the corresponding linear bundle morphism over TN which splits the exact sequence

$$0 \longrightarrow V_N TN \hookrightarrow TTN \longrightarrow TN \underset{N}{\times} TN \longrightarrow 0.$$

Definition 3. A geodesic equation on TN with respect to the connection K is defined as the image

$$\overset{\circ}{q}{}^{\mu} = \dot{q}^{\mu}, \qquad \ddot{q}^{\mu} = K^{\mu}_{\lambda} \dot{q}^{\lambda} \tag{21}$$

of the morphism (20) restricted to the diagonal $TN \subset TN \times TN$.

By a solution of a geodesic equation on TN is meant a geodesic curve $c : () \to N$, whose tangent prolongation $\dot{c}$ is an integral section (a geodesic vector field) over $c \subset N$ for the connection K. The geodesic equation (21) can be written in the form

$$\dot{q}^{\lambda} \partial_{\lambda} \dot{q}^{\mu} = K^{\mu}_{\lambda} \dot{q}^{\lambda},$$

where by $\dot{q}^{\mu}(q^{\alpha})$ is meant a geodesic vector field (which exists at least on a geodesic curve), while $\dot{q}^{\lambda}\partial_{\lambda}$ is a formal operator of differentiation (along a curve).

It is readily observed that the morphism $\widehat{K}\,|_{TN}$ is a holonomic vector field on TN. It follows that any geodesic equation (20) on TN is a second order equation on N. The converse is not true in general. Nevertheless, we have the following theorem.

Theorem 4. [7]. Every second order equation (18) on a manifold N defines a connection K_{Ξ} on the tangent bundle $TN \to N$ whose components are

$$K^{\mu}_{\lambda} = \frac{1}{2} \dot{\partial}_{\lambda} \Xi^{\mu}. \tag{22}$$

However, the second order equation (18) fails to be a geodesic equation with respect to the connection (22) in general. In particular, the geodesic equation (21) with respect to a connection K determines the connection (22) on $TN \to N$ which does not necessarily coincide with K. A second order equation Ξ on N is a geodesic equation for the connection (22) if and only if u is a spray, i.e., $[v, \Xi] = \Xi$, where $v = \dot{q}^{\lambda}\dot{\partial}_{\lambda}$ is the Liouville vector field on TN. In Section 5, we will improve Theorem 4.

4 Dynamic equations

Let $Q \to \mathbf{R}$ be a bundle coordinated by (t, q^i).

Definition 5. A second order differential equation on $Q \to \mathbf{R}$, called a dynamic equation, is defined as the image $\xi(J^1Q) \subset J^2Q$ of a holonomic connection

$$\xi = \partial_t + q^i_t \partial_i + \xi^i(t, q^j, q^j_t)\partial^t_i \tag{23}$$

on $J^1Q \to \mathbf{R}$. This is a closed subbundle of $J^2Q \to \mathbf{R}$, given by the coordinate relations

$$q^i_{tt} = \xi^i(t, q^j, q^j_t). \tag{24}$$

A solution of the dynamic equation (24), called a motion, is a curve $c : () \to Q$ whose second order jet prolongation $J^2c : () \to J^1Q$ lives in (24).

One can easily find the transformation law

$$q'^i_{tt} = \xi'^i, \qquad \xi'^i = (\xi^j \partial_j + q^j_t q^k_t \partial_j \partial_k + 2q^j_t \partial_j \partial_t + \partial^2_t) q'^i(t, q^j) \tag{25}$$

of a dynamic equation under coordinate transformations $q^i \to q'^i(t, q^j)$.

A dynamic equation ξ on a bundle $Q \to \mathbf{R}$ is said to be conservative if there exists a trivialization (1) of Q and the corresponding trivialization (2) of J^1Q such that the vector field ξ (23) on J^1Q is projectable onto M. Then this projection

$$\Xi_\xi = \dot q^i \partial_i + \xi^i(q^j, \dot q^j)\dot\partial_i$$

is a second order equation on the typical fibre M of Q. Conversely, every second order equation Ξ on a manifold M can be seen as a conservative dynamic equation

$$\xi_\Xi = \partial_t + \dot q^i \partial_i + u^i \dot\partial_i \tag{26}$$

on the bundle $\mathbf{R} \times M \to \mathbf{R}$ in accordance with the isomorphism (2).

Proposition 6. Any dynamic equation on a bundle $Q \to \mathbf{R}$ is equivalent to a second order equation on a manifold Q.

Proof. Given a dynamic equation ξ on a bundle $Q \to \mathbf{R}$, let us consider the diagram

$$\begin{tikzcd} J^2Q \arrow[r] & T^2Q \\ J^1Q \arrow[u, "\xi"] \arrow[r, "\lambda"] & TQ \arrow[u, "\Xi"'] \end{tikzcd} \tag{27}$$

where Ξ is a holonomic vector field on TQ, and we use the morphism (10). A glance at the expression (10) shows that the diagram (27) can be commutative only if the component Ξ^0 of a vector field Ξ vanishes. Since the transition functions $t \to t'$ are independent of

q^i, such a vector field may exist on TQ. Now the diagram (27) becomes commutative if the dynamic equation ξ and a vector field Ξ fulfill the relation

$$\xi^i = \Xi^i(t, q^j, \dot t = 1, \dot q^j = q^j_t). \tag{28}$$

It is easily seen that this relation holds globally because the substitution of $\dot q^i = q^i_t$ into the transformation law of a vector field Ξ restates the transformation law (25) of the holonomic connection ξ. In accordance with the relation (28), a desired vector field Ξ is an extension of the section $T\lambda \circ \lambda_2 \circ \xi$ of the bundle $TTQ \to TQ$ over the closed submanifold $J^1Q \subset TQ$ to a global section. Such an extension always exists, but is not unique. Then, the dynamic equation (24) can be written in the form

$$q^i_{tt} = \Xi^i \mid_{\dot t=1, \dot q^j = q^j_t} . \tag{29}$$

It is equivalent to the second order equation on Q

$$\ddot t = 0, \qquad \dot t = 1, \qquad \ddot q^i = \Xi^i. \tag{30}$$

Being a solution of (30), a curve c in Q also fulfills (29), and *vice versa*.

It should be emphasized that, written in the bundle coordinates (t, q^i), the second order equation (30) is well defined with respect to any coordinates on Q.

5 Dynamic connections

To say more than Proposition 6, we turn to the relationship between the dynamic equations on Q and the connections on the affine jet bundle $J^1Q \to Q$. Let

$$\gamma : J^1Q \to J^1_QJ^1Q, \qquad \gamma = dq^\lambda \otimes (\partial_\lambda + \gamma^i_\lambda \partial^t_i), \tag{31}$$

be such a connection. Its coordinate transformation law is

$$\gamma'^i_\lambda = (\partial_j q'^i \gamma^j_\mu + \partial_\mu q'^i_t) \frac{\partial q^\mu}{\partial q'^\lambda}. \tag{32}$$

Proposition 7. Any connection γ (31) on the affine jet bundle $J^1Q \to Q$ defines the holonomic connection

$$\xi_\gamma = \partial_t + q^i_t \partial_i + (\gamma^i_0 + q^j_t \gamma^i_j)\partial^t_i, \tag{33}$$

on the jet bundle $J^1Q \to \mathbf{R}$.

Proof. Let us consider the composite bundle $J^1Q \to Q \to \mathbf{R}$ and the canonical morphism ρ (13) which reads

$$\rho : J^1_QJ^1Q \ni (q^\lambda, q^i_t, q^i_{\lambda t}) \mapsto (q^\lambda, q^i_t, q^i_{(t)} = q^i_t, q^i_{tt} = q^i_{0t} + q^j_t q^i_{jt}) \in J^2Q. \tag{34}$$

A connection γ (31) and the morphism ρ (34) combine into the desired holonomic connection ξ_γ (33) on the jet bundle $J^1Q\to\mathbf{R}$.

It follows that each connection γ (31) on the affine jet bundle $J^1Q\to Q$ yields the dynamic equation

$$q^i_{tt}=(\gamma^i_0+q^j_t\gamma^i_j) \tag{35}$$

on the bundle $Q\to\mathbf{R}$. This is exactly the restriction to J^2Q of the kernel $\operatorname{Ker}\widetilde{D}_\gamma$ of the vertical covariant differential $\widetilde{D}_\gamma$ (16) defined by the connection γ:

$$\widetilde{D}_\gamma: J^1J^1Q\to V_QJ^1Q, \qquad \dot q^i_t\circ\widetilde{D}_\gamma=q^i_{tt}-\gamma^i_0-q^j_t\gamma^i_j.$$

Therefore, connections on $J^1Q\to Q$ are also called dynamic connections (one should distinguish this terminology from that of [6]). Of course, different dynamic connections may lead to the same dynamic equation (35).

Proposition 8. Any holonomic connection ξ (23) on the jet bundle $J^1Q\to\mathbf{R}$ yields the dynamic connection

$$\gamma_\xi=dt\otimes[\partial_t+(\xi^i-\frac{1}{2}q^j_t\partial^t_j\xi^i)\partial^t_i]+dq^j\otimes[\partial_j+\frac{1}{2}\partial^t_j\xi^i\partial^t_i] \tag{36}$$

on the affine jet bundle $J^1Q\to Q$.

Proof. Given an arbitrary vector field $u=a^i\partial_i+b^i\partial^t_i$ on the jet bundle $J^1Q\to\mathbf{R}$, let us put

$$I_\xi(u)=[\xi,\widehat v(u)]-\widehat v([\xi,u])=-a^i\partial_i+(b^i-a^j\partial^t_j\xi^i)\partial^t_i,$$

where $\widehat v$ is the endomorphism (4). We come to the endomorphism

$$I_\xi: VJ^1Q\underset{J^1Q}{\to}VJ^1Q,$$
$$I_\xi:\dot q^i\partial_i+\dot q^i_t\partial^t_i\mapsto -\dot q^i\partial_i+(\dot q^i_t-\dot q^j\partial^t_j\xi^i)\partial^t_i,$$

which obeys the condition $I_\xi\circ I_\xi=I_\xi$. Then there is the projection

$$J_\xi=\frac{1}{2}(I_\xi+\operatorname{Id}VJ^1Q): VJ^1Q\underset{J^1Q}{\to}V_QJ^1Q,$$
$$J_\xi:\dot q^i\partial_i+\dot q^i_t\partial^t_i\mapsto(\dot q^i_t-\frac{1}{2}\dot q^j\partial^t_j\xi^i)\partial^t_i.$$

Recall that a holonomic connections ξ on $J^1Q\to\mathbf{R}$ defines the projection

$$\widehat\xi: TJ^1Q\ni\dot t\partial_t+\dot q^i\partial_i+\dot q^i_t\partial^t_i\mapsto(\dot q^i-\dot t q^i_t)\partial_i+(\dot q^i_t-\dot\xi^i)\partial^t_i\in VJ^1Q.$$

Then the composition

$$I_\xi\circ\widehat\xi: TJ^1Q\to VJ^1Q\to V_QJ^1Q,$$
$$\dot t\partial_t+\dot q^i\partial_i+\dot q^i_t\partial^t_i\mapsto[\dot q^i_t-\dot t(\xi^i-\frac{1}{2}q^j_t\partial^t_j\xi^i)-\frac{1}{2}\dot q^j\partial^t_j\xi^i]\partial^t_i,$$

corresponds to the connection γ_ξ (36) on the affine jet bundle $J^1Q \to Q$.

The dynamic connection γ_ξ (36) possesses the property

$$\gamma_i^k = \partial_i^t \gamma_0^k + q_t^j \partial_i^t \gamma_j^k$$

which implies $\partial_j^t \gamma_i^k = \partial_i^t \gamma_j^k$. Such a dynamic connection is called symmetric.

Let γ be a dynamic connection (31) and ξ_γ the corresponding dynamic equation (33). Then the dynamic connection associated with ξ_γ takes the form

$$\gamma_{\xi_\gamma}{}_i^k = \frac{1}{2}(\gamma_i^k + \partial_i^t \gamma_0^k + q_t^j \partial_i^t \gamma_j^k), \qquad \gamma_{\xi_\gamma}{}_0^k = \xi^k - q_t^i \gamma_{\xi_\gamma}{}_i^k.$$

It is readily observed that $\gamma = \gamma_{\xi_\gamma}$ if and only if γ is symmetric.

Since the jet bundle $J^1Q \to Q$ is affine, it admits an affine connection

$$\gamma = dq^\lambda \otimes [\partial_\lambda + (\gamma_{\lambda 0}^i(q^\alpha) + \gamma_{\lambda j}^i(q^\alpha) q_t^j)\partial_i^t].$$

This connection is symmetric if and only if $\gamma_{\lambda\mu}^i = \gamma_{\mu\lambda}^i$. An affine dynamic connection generates a quadratic dynamic equation, and *vice versa*.

We use a dynamic connection in order to modify Theorem 4. Let Ξ be a second order equation on a manifold N and ξ_Ξ (26) the corresponding conservative dynamic equation on the bundle $\mathbf{R} \times N \to \mathbf{R}$. The latter yields the dynamic connection γ (36) on the bundle

$$\mathbf{R} \times TN \to \mathbf{R} \times N.$$

Its components γ_λ^μ are exactly those of the connection (22) on $TN \to N$ from Theorem 4, while γ_0^μ make up a vertical vector field

$$e = \gamma_0^\mu \dot\partial_\mu = (\Xi^\mu - \frac{1}{2}\dot q^\lambda \dot\partial_\lambda \Xi^\mu)\dot\partial_\mu \tag{37}$$

on $TN \to N$. Thus, we have proved the following.

Proposition 9. Every second order equation Ξ (18) on a manifold N admits the decomposition

$$\Xi^\mu = K_\lambda^\mu \dot q^\lambda + e^\mu$$

where K is the connection (22) on $TN \to N$, and e is the vertical vector field (37).

With a dynamic connection γ_ξ (36), one can also restate the linear connection on $TJ^1Q \to Q$, associated with a dynamic equation on Q [6] (see [5] for details).

6 Non-relativistic geodesic equations

To improve Proposition 6, we aim to show that every dynamic equation on a bundle $Q \to \mathbf{R}$ is equivalent to a geodesic equation on the tangent bundle $TQ \to Q$.

Let us consider the diagram

$$\begin{array}{ccc} J^1_Q J^1 Q & \xrightarrow{J^1\lambda} & J^1_Q TQ \\ {\scriptstyle\gamma} \uparrow & & \uparrow {\scriptstyle K} \\ J^1 Q & \xrightarrow{\lambda} & TQ \end{array} \tag{38}$$

where $J^1_Q TQ$ is the first order jet manifold of the tangent bundle $TQ \to Q$, coordinated by $(t, q^i, \dot t, \dot q^i, (\dot t)_\mu, (\dot q^i)_\mu)$, while K is a connection (19) on $TQ \to Q$.

The jet prolongation over Q of the morphism λ (3) reads

$$J^1\lambda : (t, q^i, q^i_t, q^i_{\mu t}) \mapsto (t, q^i, \dot t = 1, \dot q^i = q^i_t, (\dot t)_\mu = 0, (\dot q^i)_\mu = q^i_{\mu t}).$$

We have

$$J^1\lambda \circ \gamma : (t, q^i, q^i_t) \mapsto (t, q^i, \dot t = 1, \dot q^i = q^i_t, (\dot t)_\mu = 0, (\dot q^i)_\mu = \gamma^i_\mu),$$
$$K \circ \lambda : (t, q^i, q^i_t) \mapsto (t, q^i, \dot t = 1, \dot q^i = q^i_t, (\dot t)_\mu = K^0_\mu, (\dot q^i)_\mu = K^i_\mu).$$

It follows that the diagram (38) can be commutative only if the components K^0_μ of the connection K vanish. Since the coordinate transition functions $t \to t'$ are independent of q^i, a connection

$$\widetilde K = dq^\lambda \otimes (\partial_\lambda + K^i_\lambda \dot\partial_i) \tag{39}$$

with $K^0_\mu = 0$ may exist on $TQ \to Q$. It obeys the transformation law

$$K'^i_\lambda = (\partial_j q'^i K^j_\mu + \partial_\mu \dot q'^i)\frac{\partial q^\mu}{\partial q'^\lambda}. \tag{40}$$

Now the diagram (38) becomes commutative if the connections γ and $\widetilde K$ fulfill the relation

$$\gamma^i_\mu = K^i_\mu \circ \lambda = K^i_\mu(t, q^j, \dot t = 1, \dot q^j = q^j_t). \tag{41}$$

It is easily seen that this relation holds globally because the substitution of $\dot q^i = q^i_t$ into (40) restates the transformation law (32) of a connection on the affine jet bundle $J^1Q \to Q$. In accordance with the relation (41), a desired connection $\widetilde K$ is an extension of the section $J^1\lambda \circ \gamma$ of the affine bundle $J^1_Q TQ \to TQ$ over the closed submanifold $J^1 Q \subset TQ$ to a global section. Such an extension always exists, but is not unique. Thus, it is stated the following.

Proposition 10. In accordance with the relation (41), every dynamic equation (24) on the configuration space Q can be written in the form

$$q^i_{tt} = K^i_0 \circ \lambda + q^j_t K^i_j \circ \lambda, \tag{42}$$

where $\widetilde{K}$ is a connection (39). Conversely, each connection $\widetilde{K}$ (39) on the tangent bundle $TQ \to Q$ defines a dynamic connection γ on the affine jet bundle $J^1Q \to Q$ and the dynamic equation (42) on the configuration space Q.

Then we come to the following theorem.

Theorem 11. Every dynamic equation (24) on the configuration space Q is equivalent to the geodesic equation

$$\ddot{t} = 0, \qquad \dot{t} = 1, \qquad \ddot{q}^i = K^i_\lambda \dot{q}^\lambda, \tag{43}$$

on the tangent bundle TQ relative to a connection $\widetilde{K}$ with the components $K^0_\mu = 0$ and K^i_μ (41). Its solution is a geodesic curve in Q which also obeys the dynamic equation (42), and *vice versa.*

In accordance with this theorem, the second order equation (30) in Proposition 6 can be chosen as a geodesic equation. It should be emphasized that, written in the bundle coordinates (t, q^i), the geodesic equation (43) and the connection $\widetilde{K}$ (41) are well defined with respect to any coordinates on Q.

From the physical viewpoint, the most interesting dynamic equations are the quadratic ones

$$\xi^i = a^i_{jk}(q^\mu) q^j_t q^k_t + b^i_j(q^\mu) q^j_t + f^i(q^\mu). \tag{44}$$

This property is global due to the transformation law (25). Then one can use the following two facts.

Proposition 12. There is one-to-one correspondence between the affine connections γ on $J^1Q \to Q$ and the linear symmetric connections K (39) on $TQ \to Q$. This correspondence is given by the relation (41) which takes the form

$$\begin{aligned}
&\gamma^i_\mu = \gamma^i_{\mu 0} + \gamma^i_{\mu j} q^j_t = K_\mu{}^i{}_0(q)\dot{t} + K_\mu{}^i{}_j(q)\dot{q}^j|_{\dot{t}=1, \dot{q}^i = q^i_t} = K_\mu{}^i{}_0(q) + K_\mu{}^i{}_j(q) q^j_t, \\
&\gamma^i_{\mu\lambda} = K_\mu{}^i{}_\lambda.
\end{aligned}$$

Corollary 13. Every quadratic dynamic equation (44) gives rise to the geodesic equation

$$\begin{aligned}
&\ddot{q}^0 = 0, \qquad \dot{q}^0 = 1, \\
&\ddot{q}^i = a^i_{jk}(q^\mu)\dot{q}^j\dot{q}^k + b^i_j(q^\mu)\dot{q}^j\dot{q}^0 + f^i(q^\mu)\dot{q}^0\dot{q}^0
\end{aligned} \tag{45}$$

on TQ with respect to the symmetric linear connection

$$K_\lambda{}^0{}_\nu = 0, \quad K_0{}^i{}_0 = f^i, \quad K_0{}^i{}_j = \frac{1}{2} b^i_j, \quad K_k{}^i{}_j = a^i_{kj}. \tag{46}$$

The geodesic equation (45) however is not unique for the dynamic equation (44).

Proposition 14. Any quadratic dynamic equation (44), being equivalent to the geodesic equation with respect to the linear connection $\widetilde{K}$ (46), is also equivalent to the one with respect to an affine connection K' on $TQ \to Q$ which differs from $\widetilde{K}$ (46) in a soldering form σ on $TQ \to Q$ with the components

$$\sigma^0_\lambda = 0, \qquad \sigma^i_k = h^i_k - \frac{1}{2}h^i_k \dot{x}^0, \qquad \sigma^i_0 = -\frac{1}{2}h^i_k \dot{x}^k - h^i_0 \dot{x}^0 + h^i_0,$$

where h^i_λ are local functions on Q.

Let us extend our inspection of dynamic equations and connections to connections on the tangent bundle $TM \to M$ of the typical fibre of the configuration space $Q \to \mathbf{R}$. In this case, the relationship fails to be canonical, but depends on a trivialization (1).

Given such a trivialization, let $(t, \overline{q}^i)$ be the associated coordinates on Q, where $\overline{q}^i$ are coordinates on M with transition functions independent of t. The corresponding trivialization (2) of $J^1Q \to \mathbf{R}$ takes place in the coordinates $(t, \overline{q}^i, \dot{\overline{q}}^i)$. With respect to these coordinates, the transformation law (32) of a dynamic connection γ on the affine jet bundle $J^1Q \to Q$ reads

$$\gamma'^i_0 = \frac{\partial \overline{q}'^i}{\partial \overline{q}^j}\gamma^j_0 \qquad \gamma'^i_k = (\frac{\partial \overline{q}'^i}{\partial \overline{q}^j}\gamma^j_n + \frac{\partial \dot{\overline{q}}'^i}{\partial \overline{q}^n})\frac{\partial \overline{q}^n}{\partial \overline{q}'^k}.$$

It follows that, given a trivialization of $Q \to \mathbf{R}$, a dynamic connection γ defines the time-dependent vertical vector field

$$\gamma^i_0(t, \overline{q}^j, \dot{\overline{q}}^j)\frac{\partial}{\partial \dot{\overline{q}}^i} : \mathbf{R} \times TM \to VTM$$

and the time-dependent connection

$$d\overline{q}^k \otimes \left(\frac{\partial}{\partial \overline{q}^k} + \gamma^i_k(t, \overline{q}^j, \dot{\overline{q}}^j)\frac{\partial}{\partial \dot{\overline{q}}^i}\right) : \mathbf{R} \times TM \to J^1TM \subset TTM \tag{47}$$

on the tangent bundle $TM \to M$.

Conversely, let us consider a connection

$$\overline{K} = d\overline{q}^k \otimes \left(\frac{\partial}{\partial \overline{q}^k} + \overline{K}^i_k(\overline{q}^j, \dot{\overline{q}}^j)\frac{\partial}{\partial \dot{\overline{q}}^i}\right)$$

on the tangent bundle $TM \to M$. Given the above-mentioned trivialization of $Q \to \mathbf{R}$, the connection $\overline{K}$ defines the connection $\widetilde{K}$ (39) with the components

$$K^i_0 = 0, \qquad K^i_k = \overline{K}^i_k,$$

on the tangent bundle $TQ \to Q$. The corresponding dynamic connection γ on the affine jet bundle $J^1Q \to Q$ reads

$$\overline{\gamma}^i_0 = 0, \qquad \overline{\gamma}^i_k = \overline{K}^i_k. \tag{48}$$

Using the transformation law (32), one can extend the expression (48) to arbitrary bundle coordinates on the configuration space Q. Thus, we have proved the following.

Proposition 15. Any connection $\overline{K}$ on the typical fibre M of a bundle $Q \to \mathbf{R}$ yields a conservative dynamic equation on Q.

7 Reference frames

From the physical viewpoint, a reference frame in non-relativistic mechanics sets a tangent vector at each point of a configuration space Q which characterizes the velocity of an "observer" at this point. Thus, we come to the following geometric definition of a reference frame.

Definition 16. In non-relativistic mechanics, a reference frame is said to be a connection Γ on the bundle $Q \to \mathbf{R}$.

In accordance with this definition, the corresponding covariant differential

$$D_\Gamma(q_t^i) = q_t^i - \Gamma^i = \dot{q}_\Gamma^i$$

determines the relative velocities with respect to the reference frame Γ. In particular, given a motion c in Q, the covariant derivative $\nabla^\Gamma c$ is the velocity of this motion relative to the reference frame Γ. For instance, if c is an integral section of the connection Γ, the relative velocity of c with respect to the reference frame Γ is equal to 0. Conversely, every motion $c : \mathbf{R} \to Q$, defines a proper reference frame Γ_c such that the velocity of c relative to Γ_c equals 0. This reference frame Γ_c is an extension of the local section $J^1c : c(\mathbf{R}) \to J^1Q$ of the affine jet bundle $J^1Q \to Q$ to a global section. Such a global section always exists.

By virtue of Proposition 1, any reference frame Γ on the configuration space $Q \to \mathbf{R}$ is associated with an atlas of local constant trivializations such that $\Gamma = \partial_t$ with respect to the corresponding coordinates $(t, \overline{q}^i)$ whose transition functions are independent of time. Such an atlas is also called a reference frame. A reference frame is said to be complete if the associated connection Γ is complete. In accordance with Proposition 1 every complete reference frame provides a trivialization of a bundle $Q \to \mathbf{R}$, and *vice versa*.

Using the notion of a reference frame, we obtain a converse of Theorem 11.

Theorem 17. Given a reference frame Γ, any connection K (19) on the tangent bundle $TQ \to Q$ defines a dynamic equation

$$\xi^i = (K_\lambda^i - \Gamma^i K_\lambda^0)\dot{q}^\lambda \mid_{\dot{q}^0=1, \dot{q}^j=q_t^j}.$$

The proof follows at once from Proposition 10 and the following lemma.

Lemma 18. Given a connection Γ on the bundle $Q \to \mathbf{R}$ and a connection K on the tangent bundle $TQ \to Q$, there is the connection $\widetilde{K}$ on $TQ \to Q$ with the components

$$\widetilde{K}_\lambda^0 = 0, \qquad \widetilde{K}_\lambda^i = K_\lambda^i - \Gamma^i K_\lambda^0.$$

Let us point out the following interesting class of dynamic equations which we agree to call the free motion equations.

Definition 19. We say that the dynamic equation (24) is a free motion equation if there exists a reference frame $(t,\overline{q}^i)$ on the configuration space Q such that this equation reads

$$\overline{q}^i_{tt}=0. \tag{49}$$

With respect to arbitrary bundle coordinates (t,q^i), a free motion equation takes the form

$$q^i_{tt}=d_t\Gamma^i+\partial_j\Gamma^i(q^j_t-\Gamma^j)-\frac{\partial q^i}{\partial\overline{q}^m}\frac{\partial\overline{q}^m}{\partial q^j\partial q^k}(q^j_t-\Gamma^j)(q^k_t-\Gamma^k), \tag{50}$$

where $\Gamma^i=\partial_t q^i(t,\overline{q}^j)$ is the connection associated with the initial frame $(t,\overline{q}^i)$. One can think of the right hand side of the equation (50) as being the general coordinate expression of an inertial force in non-relativistic mechanics. The corresponding dynamic connection γ on the affine jet bundle $J^1Q\to Q$ reads

$$\gamma^i_k=\partial_k\Gamma^i-\frac{\partial q^i}{\partial\overline{q}^m}\frac{\partial\overline{q}^m}{\partial q^j\partial q^k}(q^j_t-\Gamma^j),\qquad \gamma^i_0=\partial_t\Gamma^i+\partial_j\Gamma^i q^j_t-\gamma^i_k\Gamma^k.$$

It is affine. In virtue of Proposition 12, this dynamic connection defines a linear connection K on the tangent bundle $TQ\to Q$ whose curvature is necessarily equal to 0. Thus, we come to the following criterion of a dynamic equation to be a free motion equation.

Proposition 20. If ξ is a free motion equation on a configuration space Q, it is quadratic and the corresponding linear symmetric connection (46) on the tangent bundle $TQ\to Q$ is a curvature-free connection.

This criterion fails to be a sufficient condition because it may happen that the components of a curvature-free symmetric linear connection on $TQ\to Q$ vanish with respect to the coordinates on Q which are not compatible with the fibration $Q\to\mathbf{R}$. Nevertheless, we can formulate the necessary and sufficient condition of existence of a free motion equation on a configuration space Q.

Proposition 21. A free motion equation on a bundle $Q\to\mathbf{R}$ exists if and only if the typical fibre M of Q admits a curvature-free symmetric linear connection.

Proof. Let a free motion equation take the form (49) with respect to an atlas of bundle coordinates on $Q\to\mathbf{R}$. By virtue of Proposition 8, there exists an affine dynamic connection γ on the affine jet bundle $J^1Q\to Q$ whose components relative to this atlas are equal to 0. Given a trivialization chart of this atlas, this connection defines the curvature-free symmetric linear connection on M (47). The converse statement follows at once from Proposition 15.

8 Relative acceleration

To consider a relative acceleration with respect to a reference frame Γ, one should prolong the connection Γ on $Q \to \mathbf{R}$ to a holonomic connection ξ_Γ on the jet bundle $J^1Q \to \mathbf{R}$. Note that the jet prolongation $J^1\Gamma$ (11) of Γ is not holonomic. We can construct a desired prolongation by means of a dynamic connection γ on the affine jet bundle $J^1Q \to Q$.

Let us consider the composite bundle $J^1Q \to Q \to \mathbf{R}$ and connections γ on $J^1Q \to Q$ and $J^1\Gamma$ on $J^1Q \to \mathbf{R}$. Then there exists a dynamic connection $\widetilde{\gamma}$ (17) on $J^1Q \to Q$ with the components

$$\widetilde{\gamma}^i_k = \gamma^i_k, \qquad \widetilde{\gamma}^i_0 = d_t\Gamma^i - \gamma^i_k\Gamma^k.$$

Now, let us construct some soldering form and add it to this connection. The covariant derivative of a reference frame Γ with respect to the dynamic connection γ reads

$$\nabla\Gamma = \nabla_\lambda\Gamma^k dq^\lambda \otimes \partial^t_k : Q \to T^*Q \times V_Q J^1Q, \quad \nabla_\lambda\Gamma^k = \partial_\lambda\Gamma^k - \gamma^k_\lambda \circ \Gamma. \tag{51}$$

Let us apply the canonical projection $T^*Q \to V^*Q$ and then the imbedding $\Gamma : V^*Q \to T^*Q$ to (51). We obtain the V_QJ^1Q-valued 1-form

$$\sigma = [-\Gamma^i(\partial_i\Gamma^k - \gamma^k_i \circ \Gamma)dt + (\partial_i\Gamma^k - \gamma^k_i \circ \Gamma)dq^i] \otimes \partial^t_k$$

on Q whose pull-back onto J^1Q is a desired soldering form. The sum $\gamma_\Gamma = \widetilde{\gamma} + \sigma$, called the frame connection, reads

$$\gamma_\Gamma{}^i_0 = d_t\Gamma^i - \gamma^i_k\Gamma^k - \Gamma^k(\partial_k\Gamma^i - \gamma^i_k \circ \Gamma), \qquad \gamma_\Gamma{}^i_k = \gamma^i_k + \partial_k\Gamma^i - \gamma^i_k \circ \Gamma. \tag{52}$$

This connection yields the holonomic connection

$$\xi^i_\Gamma = d_t\Gamma^i + (\partial_k\Gamma^i + \gamma^i_k - \gamma^i_k \circ \Gamma)(q^k_t - \Gamma^k).$$

Let ξ be a dynamic equation and $\gamma = \gamma_\xi$ the connection (36) associated with ξ. Then one can think of the vertical vector field

$$a_\Gamma = \xi - \xi_\Gamma = (\xi^i - \xi^i_\Gamma)\partial^i_t$$

on the affine jet bundle $J^1Q \to Q$ as being a relative acceleration with respect to the reference frame Γ.

For instance, let us consider the reference frame which is geodesic for the dynamic equation ξ, i.e.,

$$\Gamma\rfloor\nabla\Gamma = (d_t\Gamma^i - \xi^i \circ \Gamma)\partial_i = 0,$$

where $\nabla\Gamma$ is the covariant derivative (51) with respect to the dynamic connection γ_ξ. It is readily observed that integral sections c of a reference frame Γ are solutions of a dynamic equation ξ if and only if Γ is the geodesic reference frame for ξ. Then the relative acceleration of a motion c with respect to the reference frame Γ is $(\xi - \xi_\Gamma) \circ \Gamma = 0$.

Let now ξ be an arbitrary dynamic equation, written with respect to coordinates (t, q^i), proper for the reference frame Γ, i.e., $\Gamma^i = 0$. The relative acceleration with respect to the frame Γ in these coordinates is

$$a^i_\Gamma = \xi^i(t, q^j, q^j_t) - \frac{1}{2} q^k_t (\partial_k \xi^i - \partial_k \xi^i \mid_{q^j_t = 0}). \tag{53}$$

Given another bundle coordinates (t, q'^i), this dynamic equation reads

$$\xi'^i = \partial_j q'^i \xi^j (t, q^m(t, q'^k), \frac{\partial q^m}{\partial q'^k}(q'^k - \Gamma^k)) +$$
$$d_t \Gamma^i + \frac{\partial \Gamma^i}{\partial q'^j}(q'^j_t - \Gamma^j) - \partial_m q'^i \frac{\partial q^m}{\partial q'^j \partial q'^k}(q'^j_t - \Gamma^j)(q'^k_t - \Gamma^k),$$

while the relative acceleration (53) with respect to the reference frame Γ takes the form

$$a'^i_\Gamma = \partial_j q'^i a^j_\Gamma.$$

Then we can write a dynamic equation in the form, covariant under coordinate transformations:

$$\widetilde{D}_{\gamma_\Gamma} q^i_t = d_t q^i_t - \xi^i_\Gamma = a_\Gamma,$$

where $\widetilde{D}_{\gamma_\Gamma}$ is the vertical covariant differential (16) with respect to the frame connection γ_Γ (52) on $J^1Q \to Q$.

In particular, if ξ is a free motion equation which takes the form (49) with respect to a reference frame Γ, then

$$\widetilde{D}_{\gamma_\Gamma} q^i_t = 0$$

relative to any coordinates.

9 Relativistic and non-relativistic dynamic equations

In physical applications, one usually thinks of non-relativistic mechanics as being an approximation of small velocities of a relativistic theory. At the same time, the velocities in mathematical formalism of non-relativistic mechanics are not bounded. It has long been recognized that the relation between the mathematical schemes of relativistic and non-relativistic mechanics is not trivial.

Let X be a 4-dimensional world manifold of a relativistic theory, coordinated by (x^λ). Then the tangent bundle TX of X plays the role of a space of its 4-velocities. A relativistic equation of motion is said to be a geodesic equation

$$\dot{x}^\lambda \partial_\lambda \dot{x}^\mu = K^\mu_\lambda(x^\nu, \dot{x}^\nu)\dot{x}^\lambda$$

with respect to a (non-linear) connection K on $TX \to X$.

It is supposed additionally that there is a pseudo-Riemannian metric g of signature $(+,---)$ in TX such that a geodesic vector field does not leave the subbundle of relativistic hyperboloids

$$W_g = \{\dot{x}^\lambda \in TX \mid g_{\lambda\mu}\dot{x}^\lambda\dot{x}^\mu = 1\} \tag{54}$$

in TX. It suffices to require that the condition

$$(\partial_\lambda g_{\mu\nu}\dot{x}^\mu + 2g_{\mu\nu}K^\mu_\lambda)\dot{x}^\lambda\dot{x}^\nu = 0. \tag{55}$$

holds for all tangent vectors which belong to W_g (54). Obviously, the Levi–Civita connection $\{_\lambda{}^\mu{}_\nu\}$ of the metric g fulfills the condition (55). Any connection K on $TX \to X$ can be written as

$$K^\mu_\lambda = \{_\lambda{}^\mu{}_\nu\}\dot{x}^\nu + \sigma^\mu_\lambda(x^\lambda, \dot{x}^\lambda),$$

where the soldering form $\sigma = \sigma^\mu_\lambda dx^\lambda \otimes \dot{\partial}_\lambda$ plays the role of an external force. Then the condition (55) takes the form

$$g_{\mu\nu}\sigma^\mu_\lambda\dot{x}^\lambda\dot{x}^\nu = 0. \tag{56}$$

Let now a world manifold X admit a projection $X \to \mathbf{R}$, where $\mathbf{R}$ is a time axis. One can think of the bundle $X \to \mathbf{R}$ as being a configuration space of non-relativistic mechanical system. There is the canonical imbedding (3) of J^1X onto the affine subbundle

$$\dot{x}^0 = 1, \qquad \dot{x}^i = x^i_0 \tag{57}$$

of the tangent bundle TX. Then one can think of (57) as the 4-velocities of a non-relativistic system. The relation (57) differs from the familiar relation between 4- and 3-velocities of a relativistic system. In particular, the temporal component $\dot{x}^0$ of 4-velocities of a non-relativistic system equals 1 (relative to the universal unit system). It follows that the 4-velocities of relativistic and non-relativistic systems occupy different subbundles of the tangent bundle TX. Moreover, Theorem 11 shows that both relativistic and non-relativistic equations of motion can be seen as the geodesic equations on the same tangent bundle TX, but their solutions live in the different subbundles (54) and (57) of TX. At the same time, relativistic equations, expressed into the 3-velocities $\dot{x}^i/\dot{x}^0$ of a relativistic system, tend exactly to the non-relativistic equations on the subbundle (57) when $\dot{x}^0 \to 1$, $g_{00} \to 1$, i.e., where non-relativistic mechanics and the non-relativistic approximation of a relativistic theory coincide only.

Given a coordinate systems (x^0, x^i), compatible with the fibration $X \to \mathbf{R}$, let us consider a non-degenerate quadratic Lagrangian

$$L = \frac{1}{2}m_{ij}(x^\mu)x^i_0x^j_0 + k_i(x^\mu)x^i_0 + f(x^\mu), \tag{58}$$

where m_{ij} is a Riemannian mass tensor. Similarly to Proposition 12, one can show that any quadratic polynomial in $J^1X \subset TX$ is extended to a bilinear form in TX. Then the Lagrangian L (58) can be written as

$$L = -\frac{1}{2}g_{\alpha\mu}x_0^\alpha x_0^\mu, \qquad x_0^0 = 1, \tag{59}$$

where g is the metric

$$g_{00} = -2f, \qquad g_{0i} = -k_i, \qquad g_{ij} = -m_{ij}. \tag{60}$$

The corresponding Lagrange equation takes the form

$$x_{00}^i = -(m^{-1})^{ik}\{_{\lambda k\nu}\}x_0^\lambda x_0^\nu, \qquad x_0^0 = 1, \tag{61}$$

where

$$\{_{\lambda\mu\nu}\} = -\frac{1}{2}(\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu})$$

are the Christoffel symbols of the metric (60). Let us assume that this metric is non-degenerate. By virtue of Corollary 13, the dynamic equation (61) gives rise to the geodesic equation on TX

$$\dot x^\lambda \partial_\lambda \dot x^0 = 0, \qquad \dot x^0 = 1,$$
$$\dot x^\lambda \partial_\lambda \dot x^i = \{_\lambda{}^i{}_\nu\}\dot x^\lambda \dot x^\nu - g^{i0}\{_{\lambda 0\nu}\}\dot x^\lambda \dot x^\nu.$$

Let us now bring the Lagrangian (58) into the form

$$L = \frac{1}{2}m_{ij}(x^\mu)(x_0^i - \Gamma^i)(x_0^j - \Gamma^j) + f'(x^\mu), \tag{62}$$

where Γ is a Lagrangian connection on $X \to \mathbf{R}$. This connection Γ defines an atlas of local constant trivializations of the bundle $X \to \mathbf{R}$ and the corresponding coordinates $(x^0, \overline{x}^i)$ on X. In this coordinates, the Lagrangian L (62) reads

$$L = \frac{1}{2}\overline{m}_{ij}\overline{x}_0^i\overline{x}_0^j + f'(x^\mu). \tag{63}$$

One can think of its first term as the kinetic energy of a non-relativistic system with the mass tensor $\overline{m}_{ij}$ relative to the reference frame Γ, while $(-f')$ is a potential. Let us assume that f' is a nowhere vanishing function on X. Then the Lagrange equation (61) takes the form

$$\overline{x}_{00}^i = \{_\lambda{}^i{}_\nu\}\overline{x}_0^\lambda\overline{x}_0^\nu, \qquad \overline{x}_0^0 = 1, \tag{64}$$

where $\{_\lambda{}^i{}_\nu\}$ are the Christoffel symbols of the metric

$$g_{ij} = -\overline{m}_{ij}, \qquad g_{0i} = 0, \qquad g_{00} = -2f'. \tag{65}$$

This metric is Riemannian if $f' > 0$ and pseudo-Riemannian if $f' < 0$. Then the spatial part of the corresponding geodesic equation

$$\dot{\overline{x}}^\lambda \partial_\lambda \dot{\overline{x}}^0 = 0, \qquad \dot{\overline{x}}^0 = 1,$$
$$\dot{\overline{x}}^\lambda \partial_\lambda \dot{\overline{x}}^i = \{_\lambda{}^i{}_\nu\}\dot{\overline{x}}^\lambda \dot{\overline{x}}^\nu \tag{66}$$

is exactly the spatial part of the geodesic equation with respect to the Levi–Civita connection of the metric (65) on TX. It follows that the non-relativistic dynamic equation (64) describes the non-relativistic approximation of the geodesic motion in a curved space with the metric (65).

Conversely, let us consider a geodesic motion

$$\dot{x}^\lambda \partial_\lambda \dot{x}^\mu = \{_\lambda{}^\mu{}_\nu\}\dot{x}^\lambda \dot{x}^\nu \tag{67}$$

in the presence of a pseudo-Riemannian metric g on a world manifold X. Let $(x^0, \overline{x}^i)$ be local hyperbolic coordinates such that $g_{00} = 1$, $g_{0i} = 0$. These coordinates set a non-relativistic reference frame for a local fibration $X \to \mathbf{R}$. Then the equation (67) has the non-relativistic limit (66) which is the Lagrange equation for the Lagrangian (63) where $f' = 0$. This Lagrangian describes a free non-relativistic mechanical system with the mass tensor $\overline{m}_{ij} = -g_{ij}$.

In view of Proposition 14, the "relativization" (59) of an arbitrary non-relativistic quadratic Lagrangian (58) may lead to a confusion. In particular, it can be applied to a gravitational Lagrangian (62) where f' is a gravitational potential. An arbitrary quadratic dynamic equation can be written in the form

$$x^i_{00} = -(m^{-1})^{ik}\{_{\lambda k\mu}\}x^\lambda_0 x^\mu_0 + b^i_\mu(x^\nu)x^\mu_0, \qquad x^0_0 = 1,$$

where $\{_{\lambda k\mu}\}$ are the Christoffel symbols of some pseudo-Riemannian metric g, whose spatial part is the mass tensor $(-m_{ik})$, while

$$b^i_k(x^\mu)x^k_0 + b^i_0(x^\mu) \tag{68}$$

is an external force. With respect to the coordinates where $g_{0i} = 0$, one may construct the relativistic equation

$$\dot{x}^\lambda \partial_\lambda \dot{x}^\mu = \{_\lambda{}^\mu{}_\nu\}\dot{x}^\lambda \dot{x}^\nu + \sigma^\mu_\lambda \dot{x}^\lambda, \tag{69}$$

where the soldering form σ must fulfill the condition (56). It takes place only if

$$g_{ik}b^i_j + g_{ij}b^i_k = 0,$$

i.e., the external force (68) is the Lorentz-type force plus some potential one. Then, we have

$$\sigma^0_0 = 0, \qquad \sigma^0_k = -g^{00}g_{kj}b^j_0, \qquad \sigma^j_k = b^j_k.$$

The "relativization" (69) exhausts almost all familiar examples. It means that a wide class of mechanical system can be represented as a geodesic motion with respect to some affine connection in the spirit of Cartan's idea.

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